CHARACTERIZATIONS OF BANACH SPACES WHOSE DUALS ARE L_1 SPACES

BY

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ABSTRACT

We characterize those Banach spaces whose duals are isometric to L_1 spaces in terms of the structure of the spaces of absolutely summing, integral, nuclear, and fully nuclear operators from or into these spaces.

In [2] Grothendieck proved that the following are equivalent for a Banach space E:

(I) E is isometric to an $L_1(\mu)$ space;

(II) E' is a $\mathscr{P}_1$ space (see [3] for definition);

(III) For each pair of Banach spaces $F \subseteq G$, $J: E \hat{\otimes} F \to E \hat{\otimes} G$ is an isometry where J is the canonical injection. (See [1, chap. 1, §1] for definitions.)

In [4] Lindenstrauss and Lazar proved (using the terminology of [5]) that the following are equivalent for a Banach space E:

(A) E is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for every $\varepsilon > 0$;

(B) E' is an $L_1(\mu)$ space.

Our purpose here is to give other characterizations of the $\mathscr{L}_{\infty,1+\varepsilon}$ spaces in terms of classes of operators. An isomorphic version of our main result appears in [9], but we think it worthwhile to give the isometric versions without recourse to the results of [9].

We briefly define the operators with which we shall be concerned:

1) $T: E \to F$ is absolutely summing [7, chap. 2] if

$$\| T \|_{as} = \inf \left\{ C: \sum_{i=1}^{n} \| Tx_i \| \leqq C \sup_{\|x'\|=1} \sum_{i=1}^{n} | \langle x_i, x' \rangle | \right.$$
$$\left. \text{all } x_i, \cdots, x_n \in E \text{ and } n = 1, 2, \cdots \right\}$$

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is finite.

2) T: $E \to F$ is integral [1, chap. 1, §4] if the associated bilinear form on $E \times F'$ is integral, that is

$$\| T \|_{int} = \sup \left\{ \sum_{i=1}^{n} \langle Tx_i, y_i \rangle : \sup_{\substack{\| x' \| = 1 \\ \| y \| = 1}} \left| \sum_{i=1}^{n} \langle x_i, x' \rangle \langle y, y_i' \rangle \right| \leqq 1 \right\}$$

is finite.

3) $T : E \to F$ is quasi-nuclear [7, chap. 3] if there exists a sequence $\{x_i'\} \subseteq E'$ such that $\sum_{i=1}^{\infty} \| x_i' \| < +\infty$,

$$\| Tx \| \leqq \sum_{i=1}^{\infty} | \langle x, x_i' \rangle |, \text{ and } \| T \|_{qn} = \inf \left\{ \sum_{i=1}^{\infty} \| x_i' \| \right\},$$

where the infimum is taken over all $\{x_i'\}$ satisfying the above.

4) An operator $T : E \to F$ is nuclear [1] if there exist sequences

$$\{x_i'\} \subseteq E', \{y_i\} \subseteq F \text{ such that } \sum_{i=1}^{\infty} \| x_i' \| \cdot \| y_i \| < +\infty,$$

$$Tx = \sum_{i=1}^{\infty} \langle x, x_i \rangle y_i \text{ and } \| T \|_n = \inf \left\{ \sum_{i=1}^{\infty} \| x_i' \| \cdot \| y_i \| \right\},$$

where the infimum is taken over all such representations of T.

(5) An operator $T : E \to F$ is fully nuclear if the astriction $T_a : E \to \overline{T(E)}$ is nuclear and the fully nuclear norm $\| T \|_{fn}$ of T is $\| T_a \|_n$.

It should be remarked that the set of fully nuclear operators, in general, is not closed under addition [9, th. II.6].

If we denote the classes of operators defined in (1), (2), (3), (4) and (5) by $AS(E,F)$, $I(E,F)$, $QN(E,F)$, $N(E,F)$, and $FN(E,F)$, respectively, then we have the following containments

$$FN(E,F) \subseteq N(E,F) \subseteq QN(E,F) \subseteq AS(E,F)$$

$$FN(E,F) \subseteq N(E,F) \subseteq I(E,F) \subseteq AS(E,F)$$

and the natural injections are all of norm less than or equal to 1. (See [1] and [7].)

We now prove the following theorem:

THEOREM: *The following are equivalent:*

(1) *E is an $\mathscr{L}_{\infty\, 1+\varepsilon}$ space for each $\varepsilon > 0$;*

(2) *E' is isometric to an $L_1(\mu)$ space;*

(3) *$N(E,F)$ and $FN(E,F)$ are isometric for all Banach spaces F;*

(4) *$I(E,F)$ and $AS(E,F)$ are isometric for all Banach spaces F;*

(5) *$N(E,F)$ and $QN(E,F)$ are isometric for all Banach spaces F;*

(6) *$AS(F,E)$ and $I(F,E)$ are isometric for all Banach spaces F;*

(7) *$QN(F,E)$ and $N(F,E)$ are isometric for all Banach spaces F.*

REMARK: In statements (3)–(7) the word "isometric" stands for "equal as sets with the identity map being an isometry".

PROOF. We have already discussed the equivalence of (1) and (2). We begin by showing that (4) implies (5). Suppose $T : E \to F$ is quasi-nuclear, then if we inject $I: F \to l_\infty(\Gamma)$, the composition IT is nuclear and $\| IT \|_n = \| T \|_{qn}$ and we have the following factorization [7, chap. 3]

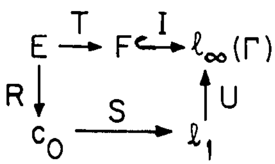

Fig. 1

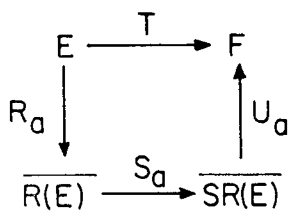

Fig. 2

where R and U are compact, $\| R \| < 1 + \varepsilon$, $\| U \| < 1 + \varepsilon$ and S is nuclear and $\| S \|_n < (1 + \varepsilon) \| IT \|_n = (1 + \varepsilon) \| T \|_{qn}$. Consider the following operators where, R_a, S_a, U_a are the astrictions of the above (Fig. 2).

Since S is nuclear, S_a is absolutely summing and $\| S_a \|_{as} \leqq \| S \|_{as} \leqq \| S \|_n$ and by (4)

$$\| S_aR_a \|_{int} = \| S_aR_a \|_{as} \leqq \| S_a \|_{as}(1 + \varepsilon)$$

Since U_a is compact and $\| U_a \| < 1 + \varepsilon$, it follows from [1, th. 10, p. 132]

$$\| T \|_n = \| U_aS_aR_a \|_n \leqq (1 + \varepsilon) \| S_aR_a \|_{int} \leqq (1 + \varepsilon)^2 \| S_a \|_{as}$$
$$\leqq (1 + \varepsilon)^2 \| S_a \|_n < (1 + \varepsilon)^3 \| T \|_{qn} \leqq (1 + \varepsilon)^3 \| T \|_n.$$

So $\| T \|_n = \| T \|_{qn}$.

To prove that (5) implies (3) we need only note that if $T: E \to F$ is nuclear, then the astriction $T_a: E \to \overline{T(E)}$ is quasi-nuclear, so by (5), $\| T_a \|_{qn} = \| T_a \|_n$, $\| T_a \|_n \geqq \| T \|_n$ and $\| T_a \|_{qn} \leqq \| T \|_n$, which establishes (3).

We now prove that (3) implies (2). Let $\{(F_\alpha, G_\alpha)\}$ be all pairs of finite dimensional spaces such that $F_\alpha \subseteq G_\alpha \subseteq l_\infty$. Let $F = (\Sigma_\alpha F_\alpha)_{l_2}$, $G = (\Sigma_\alpha G_\alpha)_{l_2}$ and let $I: F \to G$ the canonical isometry of F into G. Since F and G are reflexive with the metric approximation property, the nuclear operators from E to F and from E to G may canonically be identified with $E' \hat{\otimes} F$ and $E' \hat{\otimes} G$, respectively, The canonical operator from $E' \hat{\otimes} F$ to $E' \hat{\otimes} G$ is an isometry by (3). By the Hahn-Banach Theorem, if T is a continuous linear functional on $E' \hat{\otimes} F$ then T has an extension $\hat{T}$ to $E' \hat{\otimes} G$ such that $\| T \| = \| \hat{T} \|$. This means that given any continuous linear operator $T: F \to E''$, then there is an extension $\hat{T}: G \to E''$, $\hat{T}I = T$, such that $\| \hat{T} \| = \| T \|$. This tells us that if $A \subseteq B$ are finite dimensional spaces, and $T: A \to E''$ then there exists an extension $\hat{T}: B \to E''$ with $\| T \| = \| \hat{T} \|$. Choose Γ a large enough set so that we have an isometry $J: E'' \subseteq l_\infty(\Gamma)$. We shall show that the canonical operator from $E''' \hat{\otimes} E''$ into $E''' \hat{\otimes} l_\infty(\Gamma)$ is an isometry. Let $\Sigma_{i=1}^n z_i \otimes y_i$ be an element of $E''' \hat{\otimes} E''$, then $\Sigma_{i=1}^n z_i \otimes Jy_i$ is its canonical image in $E''' \hat{\otimes} l_\infty(\Gamma)$. There exists $w_1, \cdots, w_m \in E'''$, $\xi_1, \cdots, \xi_m \in l_\infty(\Gamma)$ such that $\Sigma_{i=1}^n z_i \otimes Jy_1 = \Sigma_{j=1}^m w_j \otimes \xi_j$, and $\Sigma_{j=1}^m \| w_j \| \, \| \xi_j \| \leqq \| \Sigma_{i=1}^n z_i \otimes Ty_i \| + \varepsilon$. Let $A = [Jy_i]_{i=1}^n$, $B \subseteq l_\infty(\Gamma)$ such that $A + [\xi_j]_{j=1}^m \subseteq B$. If we define $T: A \to E''$ by $T(a) = J^{-1}(a)$ then there exists an extension $\hat{T}: B \to E''$ with $\| \hat{T} \| = \| T \| = 1$. Then $\Sigma_{i=1}^n z_i \otimes y_i = \Sigma_{j=1}^m w_j \otimes \hat{T}\xi_j$ and $\| \Sigma_{i=1}^n z_i \otimes y_i \| \leqq \Sigma_{j=1}^m \| w_j \| \, \| \hat{T}\xi_j \| \leqq \| \Sigma_i^n z_i \otimes Jy_i \| + \varepsilon$. Thus we have an isometry, and by the Hahn-Banach Theorem, the identity operator on E''' (and element of $(E''' \hat{\otimes} E'')'$) has extension to $E''' \hat{\otimes} l_\infty(\Gamma)$. This says that E''' is isometric to a subspace of $l_\infty(\Gamma)'$ that is complemented by a projection of norm one. By Grothendieck's theorem [2], E''' and E' are isometric to L_1 spaces. We should point out that this proof is essentially that of theorem 4.1 of [6] restated in the language of tensor products.

To prove (1) implies (4), suppose $T: E \to F$ is absolutely summing. The integral norm of an operator T from E to F is the norm of T when considered as a linear functional on the space of operators (with usual operator norm) of finite rank from F to E, with the duality given by $\langle S, T' \rangle = \Sigma_{i=1}^n \langle Tx_i, y_i' \rangle$ where $Sy = \Sigma_{i=1}^n \langle y, y_i \rangle x_i'$. Suppose $Sy = \Sigma_{i=1}^n \langle y, y_i \rangle x_i'$, $\| S \| \leqq 1$. Then we shall show $| \langle S, T \rangle | < (1 + \varepsilon) \| T \|_{as}$ which will prove $\| T \|_{int} \leqq \| T \|_{as}$. Since E is an $\mathscr{L}_{\infty.1+\varepsilon}$ space there is a finite dimensional space $X \subseteq E$, $d(X, l_\infty^m) < 1 + \varepsilon$, where $m = \dim X$, and $x_i \in X$ for $1 \leqq i \leqq n$. Suppose $U: l_\infty^m \to X$ is an operator such that $\| U \| \leqq 1$, $\| U^{-1} \| < 1 + \varepsilon$, and $z_j = Ue_j$ where $\{e_j\}$ is the canonical basis of l_∞^m. Since $x_i \in X$, and $\{z_j\}$ is a basis of X, we have scalars t_{ij} such that $x_i = \Sigma_{j=1}^m t_{ij} z_j$, $1 \leqq i \leqq n$, and we have the following:

$$1 \geqq \sup_{\|y\|=1} \| Sy \| = \sup_{\substack{\|y\|=1 \\ \|x'\|=1}} | \langle Sy, x' \rangle |$$

$$= \sup \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} t_{ij} \langle z_j, x' \rangle \langle y, y_i \rangle : \| x' \| = \| y \| = 1 \right\}$$

$$= \sup \left\{ \left\| \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} t_{ij} \langle y, y_i' \rangle \right\} z_j \right\| : \| y \| = 1 \right\}$$

$$\geqq (1+\varepsilon)^{-1} \sup \left\{ \left\| \sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} t_{ij} \langle y, y_i' \rangle \right\} e_j \right\| : \| y \| = 1 \right\}$$

$$= (1+\varepsilon)^{-1} \sup_{\|y\|=1} \max_{1 \leqq j \leqq m} \left| \sum_{i=1}^{n} t_{ij} \langle y, y_i' \rangle \right|,$$

that is

$$(*) \qquad \max_{1 \leqq j \leqq m} \left\| \sum_{i=1}^{n} t_{ij} y_i' \right\| \leqq 1 + \varepsilon.$$

On the other hand, since T is absolutely summing we have:

$$\sum_{j=1}^{m} \| Tz_j \| \leqq \| T \|_{as} \sup \left\{ \sum_{j=1}^{m} | \langle z_j, x' \rangle | : \| x' \| = 1 \right\}$$

$$= \| T \|_{as} \sup \left\{ \sum_{j=1}^{m} | \langle Ue_j, x' \rangle | : \| x' \| = 1 \right\}$$

$$= \| T \|_{as} \sup \left\{ \sum_{j=1}^{m} | \langle e_j, U'x' \rangle | : \| x' \| = 1 \right\}$$

$$\leqq \| T \|_{as} \sup \left\{ \sum_{j=1}^{m} | \langle e_j, \xi \rangle | : \xi \in l_1^m, \| \xi \| = 1 \right\}$$

$$= \| T \|_{as};$$

that is

$$(**) \qquad \sum_{j=1}^{m} \| Tz_j \| \leqq \| T \|_{as}.$$

Combining inequalities (*) and (**) we have:

$$
\begin{aligned}
|\langle S,T\rangle| &= \Big|\sum_{i=1}^{n}\langle Tx_i, y_i'\rangle\Big| \\
&= \Big|\sum_{i=1}^{n}\sum_{j=1}^{m} t_{ij}\langle Tz_j, y_i'\rangle\Big| \\
&= \Big|\sum_{i=1}^{m}\langle Tz_j, \sum_{i=1}^{n} t_{ij}y_i'\rangle\Big| \\
&\leqq \sum_{j=1}^{m}\|Tz_j\|\cdot\Big\|\sum_{i=1}^{n} t_{ij}y_i'\Big\| \\
&\leqq \Big(\max_{1\leqq j\leqq m}\Big\|\sum_{i=1}^{n} t_{ij}y_i'\Big\|\Big)\sum_{j=1}^{m}\|Tz_j\| \leqq (1+\varepsilon)\|T\|_{as}
\end{aligned}
$$

which shows that $\|T\|_{int} = \|T\|_{as}$.

We have established the equivalence of (1) through (5). To facilitate the remainder of the proof we introduce

(7′) $Q(F,E)$ and $N(F,E)$ are isometric for all Banach spaces F such that F'' has the metric approximation property.

We now show that (2) implies (6). It follows from [2] that E'' is a $\mathscr{P}_1$ space; that is, for any pair of Banach spaces $F \subseteq G$, an operator $T: F \to E''$ has an extension $\tilde{T}: G \to E''$ with $\|\tilde{T}\| = \|T\|$. Suppose $T: F \to E$ is absolutely summing, then it follows from [7, chap. 3] that we may construct a factorization

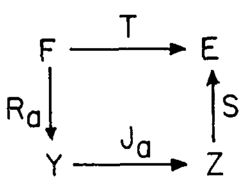

Fig. 3

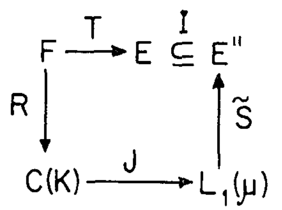

Fig. 4

where R_a is an isometry, $\|S\| \leqq \|T\|_{as}$, and J_a is the restriction and astriction of a canonical operator of the type $J: C(K) \to L_1(\mu)$, $Y \subseteq C(K)$, K a compact Hausdorff space, and μ a measure on K. Since E'' is a $\mathscr{P}_1$ space, we may extend S to an operator $\tilde{S}$ defined on $L_1(\mu)$ into E'' and obtain the factorization (Fig. 4) with $\|T\|_{int} = \|IT\|_{int} = \|\tilde{S}JR\|_{int} \leqq \|\tilde{T}\|\cdot\|J\|_{int} = \|\tilde{S}\| \leqq \|S\| \leqq \|T\|_{as} \leqq \|T\|_{int}$.

To prove (6) implies (7′), let F be a Banach space such that F'' has the metric approximation property and let $T: F \to E$ be a quasi-nuclear operator and let

$I: E \to l_\infty(\Gamma)$ be the canonical injection. The IT is nuclear, $\| IT \|_n = \| T \|_{fn}$, and we have the factorization of [7, chap. 3],

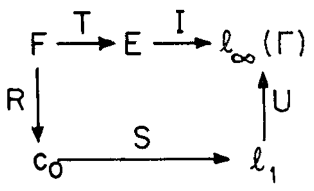

Fig. 5

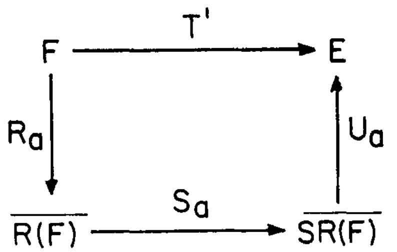

Fig. 6

with $\| R \| < 1 + \varepsilon$, $\| U \| < 1 + \varepsilon$, $\| S \|_n < (1 + \varepsilon) \| IT \|_n$.

From this we obtain the above factorization (Fig. 6).

Since S_a is absolutely summing, $U_a S_a$ is absolutely summing and by hypothesis $U_a S_a$ is integral and $\| U_a S_a \|_{as} = \| U_a S_a \|_{int}$. Since R_a is compact and $\| R_a \| < 1 + \varepsilon$, we have [1. lemma 14, p. 133] that

$$\begin{aligned} \| R'_a S'_a U'_a \|_n &\leqq \| R'_a \| \cdot \| S'_a U'_a \|_{int} < (1 + \varepsilon) \| U_a S_a \|_{int} \\ &= (1 + \varepsilon) \| U_a S_a \|_{as} \leqq (1 + \varepsilon) \| S \|_{as} \| U \| \\ &\leqq (1 + \varepsilon)^2 \| S \|_n < (1 + \varepsilon)^3 \| IT \|_n = (1 + \varepsilon)^3 \| T \|_{qn}. \end{aligned}$$

Thus $\| T' \|_n \leqq (1 + \varepsilon)^3 \| T \|_{qn}$ and since F'' has the metric approximation property, $\| T \|_n = \| T' \|_n$ so $\| T \|_n = \| T \|_{qn}$.

In proving $(7') \to (2)$, let $F = (\Sigma_\alpha \oplus E_\alpha)_{l_2}$, $\{E_\alpha\}$ the finite dimensional subspaces of l_∞. Since l_∞ isometrically contains all separable Banach spaces, F has as subspaces, complemented by projections of norm 1, every finite dimensional Banach space. Also, F is reflexive and has the metric approximation property. If $I: E \to G$ is an isometry into a Banach space G, and $T: F \to E$ is an operator such that IT is nuclear then T is quasi-nuclear, $\| T \|_{qn} \leqq \| IT \|_n$, and $\| T \|_n = \| T \|_{qn}$ by $(7')$. Since $\| IT \|_n \leqq \| T \|_n$ this proves $\| T \|_n = \| IT \|_n$; that is, the space of nuclear operators from F to E is isometric (under composition) to a subspace of the nuclear operators from F to G. Since F' has the approximation property, from the duality theorem [1], the restriction operator from $\mathscr{L}(G, F)$ to $\mathscr{L}(E, F)$ is onto and thus if $S \in \mathscr{L}(E, F)$, there exists an element $\tilde{S} \in \mathscr{L}(G, F)$, $\| \tilde{S} \| < (1 + \varepsilon) \| S \|$, such that S is the restriction of $\tilde{S}$ to E. What we have proved is this: if E_α is finite dimensional, $E \subseteq G$, $T: E \to E_\alpha$ is an operator, then there exists $\tilde{T}: G \to E_\alpha$ an extension of

T such that $\| \tilde{T} \| < (1+\varepsilon) \| T \|$. By repeating the argument of theorem 4.1 of [6], with obvious modifications, we have that E' is isometric to a complemented (by a projection of norm 1) subspace of an $L_1(\mu)$ space hence is an $L_1(\nu)$ space for some measure ν [2].

Since (7) implies (7′) is obvious we have only to show (1) implies (7). Suppose E is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for each $\varepsilon > 0$, and $T\colon F \to E$ is quasi-nuclear. If $I\colon E \to l_\infty(\Gamma)$ is the canonical injection, then the composition IT is nuclear, $\| IT \|_n = \| T \|_{qn}$ and we again have the factorization [7]:

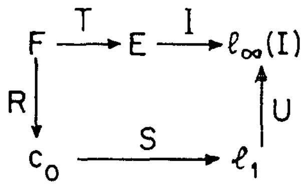

Fig. 7

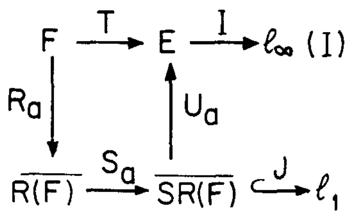

Fig. 8

with U and R compact, $\| R \| < 1+\varepsilon$, $\| U \| < 1+\varepsilon$, and $\| S \|_n < (1+\varepsilon) \| IT \|_n$. From this we obtain the above factorization (Fig. 8). Since $U_a\colon \overline{SR(F)} \to E$ is compact E is an $\mathscr{L}_{\infty,1+\varepsilon}$ space for each $\varepsilon > 0$, then there is an extension $\tilde{U}_a\colon l_1 \to E$ with $\| \tilde{U}_a \| < (1+\varepsilon) \| U_a \|$ [3]. Since JS_a is nuclear and $\| JS_a \|_n \leqq \| S \|_n$, we have that $\| T \|_n = \| \tilde{U}_a J S_a R_a \|_n \leqq \| U_a \| \; \| JS_a \|_n \; \| R_a \| < (1+\varepsilon)^2 \; \| S \|_n \; (1+\varepsilon) < (1+\varepsilon)^4 \| IT \|_n = (1+\varepsilon)^4 \| T \|_{qn}$. This proves $\| T \|_n = \| T \|_{qn}$ and hence (1) implies (7).

The following corollaries follow immediately from the theorem.

COROLLARY. 1. *Suppose $T\colon E \to F$ is an operator of finite rank. If F is an $\mathscr{L}_{\infty,1+\varepsilon}$ space, then $\| T \|_{as} = \| T \|_n = \| T \|_{int}$. If E is an $\mathscr{L}_{\infty,1+\varepsilon}$ space, all the above norms are equal to the fully-nuclear norm. The converse is also true, and follows easily from* [2].

COROLLARY 2. (Pietsch [8]) *Suppose $T\colon E \to F$ is of finite rank, E, F Banach spaces, then $\| T \|_{as} = \| T \|_{qn}$. (Which follows from the above corollary by considering F as a subspace of some $l_\infty(\Gamma)$).*

It is possible for us to give a proof of the following theorem of D. R. Lewis [10]:

THEOREM. *Let E be a Banach space. Then E' is isometric to $l_1(\Gamma)$ if and only if $AS(E,F)$ and $N(E,F)$ are isometric for all Banach spaces F.*

PROOF. Suppose E' is isometric to $l_1(\Gamma)$. Let $T\colon E \to F$ be absolutely summing. By the theorem above, $\| T \|_{as} = \| T \|_{int}$. Since the second conjugate of E has the metric approximation property, we have $\| T' \|_{int} = \| T \|_{int}$ and $\| T' \|_n = \| T \|_n$ [1, prop. 15]. By theorem 11 of [1] we know $\| T' \|_n = \| T' \|_{int}$. If $AS(E,F)$ and $N(E,F)$ are isometric, then $I(E,F)$ and $N(E,F)$ are isometric so E' is isometric to some $L_1(K,\mu)$ which we may assume to be a positive Radon measure μ on some locally compact Hausdorff space K. If μ is not purely atomic, then there exist a measurable set $A \subseteq K$, $\mu(A) = c > 0$ and sets $\{A_{ni}\}_{n=1}^{\infty}{}_{i=1}^{2^n}$ such that $\cup_{i=1}^{2^n} A_{ni} = A$, $\mu(A_{ni}) = 2^{-n}c$. If x'_{ni} is the element of E' corresponding to $\chi_{A_{ni}} d\mu$ then $\| x'_{ni} \| = 2^{-n}c$. Define $T : E \to c_0$ such that $Tx = \{x'_{ni}(x)\}_{n=1}^{\infty}{}_{i=1}^{2n}$. The adjoint $T' : l_1 \to E'$ sends the unit ball of l_1 into the convex, balanced hull of $\{x_{ni}\}_{n=1}^{\infty}{}_{i=1}^{2^n}$ which, regarded as a subset of $L_1(K,\mu)$, is lattice bounded but not equimeasurable. By [1, prop. 9, p. 64] we have that T is integral but not nuclear, thus T is compact and integral but not nuclear, which is a contradiction. Thus $L_1(K,\mu)$ is purely atomic.

It is possible to use the above argument and the theorem of Zippin [11] that each infinite dimensional $\mathscr{L}_{\infty,1+\varepsilon}$ space contains a subspace isometric to c_0, to give a proof of the following observation of D. R. Lewis: Let E be a Banach space such that $AS(F,E)$ and $N(F,E)$ are isometric for each Banach space F. Then E is finite dimensional. Since E must be an $\mathscr{L}_{\infty,1+\varepsilon}$ space, if E is infinite dimensional, there exists a subspace $G \subseteq E$, G isometric to c_0. Let F be any Banach space such that F' is isometric to $L_1(\mu)$, with μ not purely atomic. Construct $T: F \to G$ as above such that T is integral but not nuclear. If $I: F \to E$ denotes containment, then IT is not nuclear unless T is since G is an $\mathscr{L}_{\infty,1+\varepsilon}$ space. Thus $IT: F \to E$ is integral but not nuclear. Thus E is finite dimensional.

It would be very interesting to know what Banach spaces E have the property (i) that $N(F,E)$ and $I(F,E)$ are isometric (or even isomorphic) for every Banach space F or (ii) $N(E,F)$ and $I(E,F)$ are isometric for all F. If E is a reflexive Banach space with the metric approximation property, then E satisfies both (i) and (ii) [1, theorem 10]. If E is separable conjugate space with the metric approximation property, then E satisfies (ii) [1, theorem 10]. It would be interesting to know if there is a Banach space E that contains no subspace isomorphic to l_1, such that $N(E,F)$ is a proper subset of $I(E,F)$ for some F.

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